UDC 531.8

ON THE PROBLEM OF STANDING OF A PACING APPARATUS

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A special case of the problem of two-leg pacing, namely that of standing on one fixed foot [1], is investigated. Dynamics of the torso are studied on the assumption that the leg support and attachment points are fixed in space.

1. The equations of motion. We consider the motion of the torso standing on one leg in the stationary system of coordinates $NXYZ$ (Fig. 1) in which N is the coordinate origin and the Z-axis is directed vertically upward. We assume the torso to be a solid body of weight $P = Mg$ (M is the mass of body and g the acceleration of gravity). Two multilink legs consisting of weightless noninertial links connected by three-stage joints are attached to the body at point O. The body whose center of mass is at point C is supported on one of the legs. We denote by $\mathbf{r}_C$, $\mathbf{r}_O$, and $\mathbf{r}_\nu$ the position vectors drawn from N to the body center of mass, the attachement point O, and the support point, respectively, and by $\boldsymbol{\rho}$ the position vector OC. Then $\mathbf{r}_C = \mathbf{r}_O + \boldsymbol{\rho}$. We denote by $\mathbf{P}$ the gravity force vector and by $\mathbf{R}$ the resultant vector of the support reaction force which includes the normal component of reaction and the friction force, i. e. the adhesion.

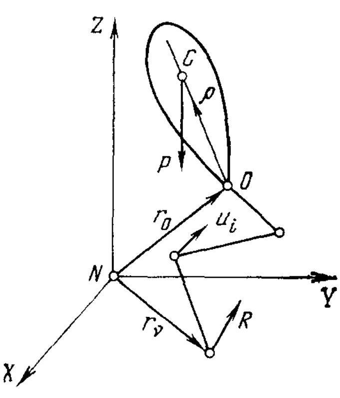

Fig. 1

We introduce in the analysis the reference point $\mathbf{e}_X$, $\mathbf{e}_Y$, $\mathbf{e}_Z$ of unit vectors along axes X, Y, and Z and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ the reference point of unit vectors along axes $Ox'y'z'$ rigidly attached to the body, which we assume to coincide with the principal axes of inertia of the body at point O. The relative position of axes is determined by the matrix of directional cosines

$$\mathbf{e}_X \cdot \mathbf{e}_k = \alpha_k, \quad \mathbf{e}_Y \cdot \mathbf{e}_k = \beta_k, \quad \mathbf{e}_Z \cdot \mathbf{e}_k = \gamma_k \quad (k = 1,\ 2,\ 3) \tag{1.1}$$

The equations of motion of the system which represent the theorem on the change of its momentum and the theorem on its moment of momentum relative to point O can be presented in the form [1]

$$\mathbf{R} = -\mathbf{P} + M\{\mathbf{r}_O^{\cdot\cdot} + \boldsymbol{\omega}^{\cdot} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times [\boldsymbol{\omega} \times \boldsymbol{\rho}]\} \tag{1.2}$$
$$J\boldsymbol{\omega}^{\cdot} + \boldsymbol{\omega} \times J\boldsymbol{\omega} + M\boldsymbol{\rho} \times \mathbf{r}_O^{\cdot\cdot} = \boldsymbol{\rho} \times \mathbf{P} - (\mathbf{r}_O - \mathbf{r}_\nu) \times \mathbf{R}$$

where $\boldsymbol{\omega}$ is the angular velocity vector of the body and J is the body inertia tensor at point O. The symbol $\boldsymbol{\omega}^{\cdot}$ denotes the dervative of angular velocity of the body in fixed axes, so that

$$\frac{d\mathbf{L}}{dt} = J\boldsymbol{\omega}^{\cdot} + \boldsymbol{\omega} \times J\boldsymbol{\omega} \tag{1.3}$$

where $\mathbf{L}$ is the moment of momentum vector. We assume that control moments $\mathbf{u}_i$ are applied at each i-th of the leg joints. Owing to the noninertial properties of legs, the application of the theorem on the change of the moment of momentum relative to the i-th joint to the part of leg between the support point and the i-th joint yields

$$\mathbf{u}_i = (\mathbf{r}_i - \mathbf{r}_\nu) \times \mathbf{R}$$

where $\mathbf{r}_i$ is the position vector of the i-th joint drawn from the origin N of the stationary coordinate system.

The last term in the second of Eqs. (1.2), thus, represents the moment applied to the body. It depends on the reaction force $\mathbf{R}$ which may be considered as the control. In conformity with the statement of the problem this control ensures the immobility of the point of leg attachment. The point of support is also fixed. Hence

$$\mathbf{r}_O = \text{const}, \quad \mathbf{r}_\nu = \text{const} \tag{1.4}$$

Generally $\mathbf{r}_O = \mathbf{r}_O(t)$ is a given continuous function and $\mathbf{r}_\nu = \mathbf{r}_\nu(t)$ a given piecewise continuous function. Equations (1.2) then define the process of the so-called single-support step [1].

On the strength of (1.4) we have in Eqs.(1.2) $\mathbf{r}_O^{\cdot\cdot} = 0$. Taking this into account and using the first of Eqs. (1.2), we eliminate from the second the reaction $\mathbf{R}$ and obtain for the motion of the body the equation

$$J\boldsymbol{\omega}^{\cdot} + \boldsymbol{\omega} \times J\boldsymbol{\omega} = \boldsymbol{\rho} \times \mathbf{P} + \mathbf{r} \times \mathbf{P} - M\mathbf{r} \times \{\boldsymbol{\omega}^{\cdot} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times [\boldsymbol{\omega} \times \boldsymbol{\rho}]\} \tag{1.5}$$

$$\mathbf{P} = -P\mathbf{e}_Z, \quad \mathbf{r} = x\mathbf{e}_X + y\mathbf{e}_Y + z\mathbf{e}_Z$$

$$\boldsymbol{\rho} = x_O'\mathbf{e}_1 + y_O'\mathbf{e}_2 + z_O'\mathbf{e}_3, \quad \boldsymbol{\omega} = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2 + \omega_3\mathbf{e}_3$$

$$J = \operatorname{diag}\{A,\ B,\ C\}$$

where $\mathbf{r} = \mathbf{r}_O - \mathbf{r}_\nu$ is a vector constant in space whose introduction is tantamount to the transfer of the origin N of the stationary coordinate system to the support point, and x, y, z, x_O', y_O', and z_O' are constant quantities. The first term in the right-hand side of (1.5) is the moment of the force of gravity and the remaining terms represent the control moment. It is apparent that the control moment depends not only on phase coordinates and velocities but, also, on accelerations. System (1.5) is closed by the Poisson kinematic equations

$$\mathbf{e}_X^{\cdot} = \mathbf{e}_X \times \boldsymbol{\omega}, \quad \mathbf{e}_Y^{\cdot} = \mathbf{e}_Y \times \boldsymbol{\omega}, \quad \mathbf{e}_Z^{\cdot} = \mathbf{e}_Z \times \boldsymbol{\omega} \tag{1.6}$$

Let us pass to the investigation of system (1.5), (1.6) whose particular case of $\mathbf{r} = 0$ is the classical problem of dynamics of a heavy solid body. Hence it is reasonable also in the case of $\mathbf{r} \neq 0$ to consider, first of all, the transformation used in well studied cases (e.g., integrable) of problems of heavy body motions.

Below, we consider transformations in three classical problems, namely those of Euler, Lagrange, and of plane motion.

2. Transformation in the case of Euler's problem [1]. Let the leg attachment point be at the body center of mass. Then $\boldsymbol{\rho} = 0$ and

Eq. (1.5) with allowance for (1.3) is conveniently written in the form

$$d\mathbf{L}/dt = \mathbf{r} \times \mathbf{P} \qquad (2.1)$$

The control moment vector $\mathbf{r} \times \mathbf{P}$ applied to the body is constant and lies in the horizontal plane, which means that the moment of momentum $\mathbf{L}$ increases indefinitely. We call such motion "unlimited unwiding of the body". For the components L_X, L_Y, and L_Z of vector $\mathbf{L}$ we have the following first integrals:

$$L_X = -Pyt + C_1, \quad L_Y = Pxt + C_2, \quad L_Z = C_3$$

where C_1, C_2, and C_3 are constants of integration. The direction of vector $\mathbf{L}$ asymptotically approaches the direction of the control moment acting in the horizontal plane.

We recall that the considered system resting on one and the same point of the surface retains the foot immobility in any arbitrary position, and the system center of mass cannot at any time lie on the vertical line passing through the support point. The system supported on the surface on a single point of the foot does not fall, although standing obliquely. This is achieved owing to the infinite increase of the moment of momentum. (Thus a man standing on his heels tries to maintain equilibrium by rotating his arms). Although this case had been already described in [1], it is presented here for the sake of making subsequent exposition understandable.

As implied by (2.1), infinite unwinding of the system does not occur only when $\mathbf{r} \| \mathbf{P}$ and, then, problem (2.1) degenerates to the classical case of Euler's problem.

Infinite unwinding of the body can also occur when $\boldsymbol{\rho} \neq 0$, since then the constant (in space) vector $\mathbf{r} \times \mathbf{P}$ additionally appears in the moment applied to the body.

It is interesting to note that bounded motions of the body also exist. This will be shown below.

3. Transformation in the case of the Lagrange problem. We impose on parameters in (1.5) the following conditions:

$$\mathbf{r} = h\mathbf{e}_Z, \quad A = B, \quad \boldsymbol{\rho} = \rho\mathbf{e}_3 \qquad (3.1)$$

the first of which means that the point of leg attachment is located exactly over the support point $(x = y = 0,\ z = h)$. The last two conditions in (3.1) are Lagrange conditions: the body is dynamically symmetric $(A = B)$ and the center of mass lies on the axis $(x_O' = y_O' = 0,\ z_O' = \rho)$ of dynamic symmetry.

Let us consider the problem of the first integrals of Eqs. (1.5) and (1.6) under conditions (3.1). Since $\mathbf{P} = -P\mathbf{e}_Z$ and $\mathbf{r} = h\mathbf{e}_Z$, the resultant moment vector in the right-hand side of (1.5) of the normal to $\mathbf{e}_Z$, which by virtue of (1.3) (and when $A \neq B \neq C$) leads to the integral

$$L_Z = K_1 \qquad (3.2)$$

which in the case of $A = B$ is of the form

$$A\ (\omega_1\gamma_1 + \omega_2\gamma_2) + C\omega_3\gamma_3 = K_1 \qquad (3.3)$$

We substitute now expressions (3.1) and (3.2) into (1.5) and carry out scalar multiplication of both of its sides by $\mathbf{e}_3$. Taking into account formula $\mathbf{a} \times [\mathbf{b} \times \mathbf{c}] = \mathbf{b}\ (\mathbf{ac}) - \mathbf{c}\ (\mathbf{ab})$, after transformation we obtain

$$C\omega_3^{\cdot} = M\rho h \ \{(\mathbf{e}_Z\boldsymbol{\omega}^{\cdot}) - (\boldsymbol{\omega}^{\cdot}\mathbf{e}_3)(\mathbf{e}_Z\mathbf{e}_3) - (\mathbf{e}_3\boldsymbol{\omega}) \ ([\mathbf{e}_Z \times \boldsymbol{\omega}] \ \mathbf{e}_3)\} \qquad (3.4)$$

Taking into account the third of Eqs. (1.6), we can verify that the expression within braces represents the total derivative of the expression

$$(\mathbf{e}_Z\boldsymbol{\omega}) - (\mathbf{e}_Z\mathbf{e}_3) \ (\boldsymbol{\omega}\mathbf{e}_3) = \gamma_1\omega_1 + \gamma_2\omega_2$$

Hence integration of (3.4) yields the new first integral

$$C\omega_3 - M\rho h \ (\gamma_1\omega_1 + \gamma_2\omega_2) = K_2 \qquad (3.5)$$

which for $h = 0$ becomes the Lagrange integral.

In the investigation the following relation between integrals (3.3) and (3.5) may also prove useful:

$$(aA - bM\rho h)(\gamma_1\omega_1 + \gamma_2\omega_2) + C\omega_3(a\gamma_3 + b) = aK_1 + bK_2 \qquad (3.6)$$

where a and b are arbitrary constants. When $aA = bM\rho h$ we obtain from (3.6) the integral

$$\omega_3(A + M\rho h\gamma_3) = C_1 \qquad (3.7)$$

If in (3.6) $Ca = aA - bM\rho h$, that integral assumes the form

$$M\rho h \ (\boldsymbol{\omega}\mathbf{e}_Z) + (A - C) \ \omega_3 = C_2 \qquad (3.8)$$

Integral (3.8) is necessary for deriving the energy integral which also exists in this case. Let us prove this.

Carrying out the usual procedure for obtaining the energy integral and taking into account conditions (1.6) and (3.1), we find that

$$\frac{1}{2}\frac{d}{dt}(\boldsymbol{\omega}J\boldsymbol{\omega}) = -P\rho\frac{d}{dt}(\mathbf{e}_3\mathbf{e}_Z) - M\rho h\,(\mathbf{e}_3\mathbf{e}_Z)\frac{1}{2}\frac{d\omega^2}{dt} + \qquad (3.9)$$
$$M\rho h\,(\mathbf{e}_3\boldsymbol{\omega})(\mathbf{e}_Z\boldsymbol{\omega}^{\cdot}) + M\rho h\omega^2([\boldsymbol{\omega} \times \mathbf{e}_Z]\,\mathbf{e}_3)$$

But by virtue of (3.1) and (3.8)

$$\frac{1}{2}\frac{d}{dt}(\boldsymbol{\omega}J\boldsymbol{\omega}) = \frac{1}{2}A\frac{d}{dt}\omega^2 + (C - A)(\mathbf{e}_3\boldsymbol{\omega})(\mathbf{e}_3\boldsymbol{\omega}^{\cdot}) = \qquad (3.10)$$
$$\frac{1}{2}A\frac{d}{dt}\omega^2 + M\rho h\,(\mathbf{e}_Z\dot{\boldsymbol{\omega}})(\mathbf{e}_3\boldsymbol{\omega})$$

Substituting (3.10) into (3.9) and taking into account (1.6), we reduce (3.9) to the form

$$\frac{1}{2}\frac{d}{dt}A\omega^2 + P\rho\frac{d}{dt}(\mathbf{e}_3\mathbf{e}_Z) + M\rho h\,(\mathbf{e}_3\mathbf{e}_Z)\frac{1}{2}\frac{d\omega^2}{dt} + \qquad (3.11)$$
$$M\rho h\omega^2\left(\mathbf{e}_3\frac{d\mathbf{e}_3}{dt}\right) = 0$$

where in conformity with (1.1) $(\mathbf{e}_3\mathbf{e}_Z) = \gamma_3$, $\mathbf{e}_3 d\mathbf{e}_Z / dt = d\gamma_3 / dt$. Relation (3.11) is made integrable by multiplying it by the integrating factor $A + M\rho h\gamma_3$. After integration we obtain the following first integral

$$\left(\omega^2 + \frac{g}{h}\right)(A + M\rho h\gamma_3)^2 = E \qquad (3.12)$$

where E is the constant of integration. Integral (3.12) is a transform of the energy integral in the case of the Lagrange problem. This can be readily verified by passing in (3.12) (with allowance for (3.5)) to limit $h \to 0$.

It is interesting that (3.12) and (3.7) yield, as a corollary, a first integral that contains only angular velocity components

$$(\omega^2 + gh^{-1}) / \omega_3^2 = E / c_1^2$$

We have, thus, obtained three independent first integrals, viz. (3.3), (3.5), and (3.12) which together with the fourth (trivial) integral

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \tag{3.13}$$

enable us to reduce the problem to quadratures, namely

$$\left(\frac{d\gamma_3}{dt}\right)^2 = (1 - \gamma_3^2)\left\{\frac{C^2E - (M\rho hK_1 + AK_2)^2}{C^2(A + M\rho h\gamma_3)^2} - \frac{g}{h}\right\} - \left(\frac{K_1 - K_2\gamma_3}{A + M\rho h\gamma_3}\right)^2$$

Hence

$$\frac{d\gamma_3}{dt} = \pm\frac{\sqrt{P_4(\gamma_3)}}{A + M\rho h\gamma_3} \tag{3.14}$$

where $P_4(\gamma_3)$ is a fourth power polynomial of the indicated argument. The inversion $\gamma_3(t)$ makes possible the calculation in the usual manner of Euler's angles which define the position of the body.

Without going into the quantitative definition of the motion, we shall consider its qualitative properties. For this we investigate the roots of polynomial

$$P_4(\gamma_3) = (1 - \gamma_3^2)\left\{E - \frac{(M\rho hK_1 + AK_2)^2}{C^2} - \frac{g}{h}(A + M\rho h\gamma_3)^2\right\} - (K_1 - \gamma_3K_2)^2$$

It is evident that

$$P_4(\gamma_3) \sim M^2\rho^2hg\,\gamma_3^4 \quad \text{as} \quad \gamma_3 \to \infty$$

$$P_4(\gamma_3 = \pm 1) = -(K_1 \pm K_2)^2 \leqslant 0$$

Thus, when $\rho h \neq 0$ the polynomial $P_4(\gamma_3)$ has in the interval $\gamma_3 \in (-1, 1)$ two roots which we denote by u_1 and u_2, and assume that $u_1 < u_2$.

Let us first assume that the sign of the denominator in formula (3.14) does not change in the admissible variation interval $\gamma_3 \in [u_1, u_2]$. As in [2], we consider the qualitative properties of motion of the end of the Oz'-axis over an imaginary unit sphere rigidly attached to the system of axes $OXYZ$. The Oz'-axis oscillates in the spherical layer defined by u_1 and u_2 (Fig. 2).

To determine the relative position of systems $OXYZ$ and $Ox'y'z'$ we introduce Euler's angles ψ, θ, φ [2]. Then

$$\omega_1 = \psi^{\cdot}\sin\theta\sin\varphi + \theta^{\cdot}\cos\varphi, \quad \omega_2 = \psi^{\cdot}\sin\theta\cos\varphi - \theta^{\cdot}\sin\varphi$$

$$\gamma_1 = \sin\theta\sin\varphi, \quad \gamma_2 = \sin\theta\cos\varphi, \quad \gamma_3 = \cos\theta$$

From which

$$\gamma_1\omega_1 + \gamma_2\omega_2 = \psi^{\cdot}(1-\gamma_3{}^2)$$

Using formulas (3.3) and (3.5) we finally obtain

$$\psi^{\cdot} = \frac{1}{1-\gamma_3{}^2}\,\frac{K_1 - K_2\gamma_3}{A + M\rho h\gamma_3}$$

We denote by χ and w the angles formed by the meridional arc $z'Z$ with arc ZX and the tangent to the trajectory of point z' respectively. Then

$$\chi = \psi - \frac{\pi}{2}, \quad \operatorname{tg} w = \frac{\sin\theta d\chi}{d\theta} = \frac{-(1-\gamma_3{}^2)\,d\chi}{d\gamma_3} = \pm\frac{K_1 - K_2\gamma_3}{\sqrt{P_4(\gamma_3)}} \qquad (3.15)$$

Hence the trajectory of point z' has a monotonic character (Fig. 2, a), provided function $(K_1 - K_2\gamma_3)$ does not change its sign for $\gamma_3 \in [u_1, u_2]$. In the opposite case the trajectory is loop-shaped (Fig. 2, b). Finally, when $(K_1 - K_2\gamma_3)$ vanishes at the ends $\gamma_3 = u_1$ or $\gamma_3 = u_2$ of the segment, the trajectory has cusps (Fig. 2, c).

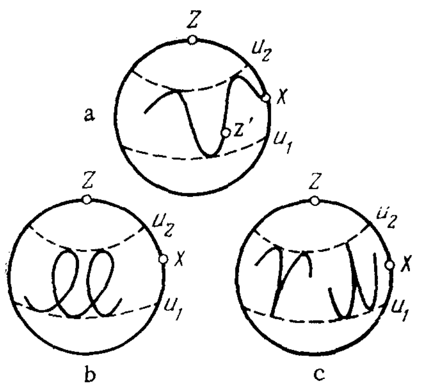

Fig. 2

Let now $A + M\rho h u' = 0$ where $u' \in (u_1, u_2)$. The qualitative pattern of motion of the Oz'-axis end over the unit sphere remains unchanged, since the form of formula (3.15) does not change. However, in conformity with (3.14), the angular velocity $d\gamma_3 / dt$ (and, generally, also the angular velocity of Euler's angles) attain infinitely large values when crossing the parallel $\gamma_3 = u'$. At the instant of crossing an effect similar to a shock produced by the support is observed. A particular case of this effect is considered below in the investigation of the plane problem.

4. **Stability of vertical rotation.** Equations (1.5) and (1.6) with consitions (3.1) have a particular solution that corresponds to rotation of the body at constant angular velocity about the vertically oriented dynamic symmetry axis

$$\gamma_3 = 1, \ \gamma_1 = \gamma_2 = 0, \ \omega_1 = \omega_2 = 0, \ \omega_3 = r_0 \qquad (4.1)$$

Let us analyze the stability of (4.1). The perturbed motion

$$\omega_1 = p, \ \omega_2 = q, \ \omega_3 = r_0 + \xi; \quad \gamma_1 = \gamma, \ \gamma_2 = \gamma', \ \gamma_3 = 1 + \delta \qquad (4.2)$$

has the following first integrals

$$\begin{aligned} V_1 &= 2A_0{}^2 r_0\xi + 2M\rho A_0 g_0\delta + A_0{}^2(p^2 + q^2 + \xi^2) + \\ &\quad M^2\rho^2 h g_0\delta^2 + 4M\rho h r_0 A_0\xi\delta + \ldots \\ V_2 &= A_0(p\gamma + q\gamma') + C\xi\delta + Cr_0\delta \\ V_3 &= \gamma^2 + \gamma'^2 + \delta^2 + 2\delta = 0 \\ V_4 &= A_0 C\xi + M\rho h C\xi\delta + M\rho h C r_0\delta \end{aligned} \qquad (4.3)$$

$$A_0 = A + M\rho h, \quad g_0 = g + hr_0{}^2 \qquad (4.4)$$

Integrals V_1 and V_3 are obtained from formulas (3.12) and (3.14). Integrals V_2 and V_4 are, respectively the rests of integrals (3.3) and (3.5), and the corollary of formula (3.7). Only terms of the order of smallness not higher than the second are retained, since higher order terms are unimportant for stability investigation.

We use the method of Chetaev [3] for determining the Liapunov function of the form of a quadratic sheef of integrals (4.3)

$$L = V_1 + 2\varkappa V_2 + \alpha V_3 + \beta V_4 + kV_4^2 + mV_3\delta \qquad (4.5)$$

Setting

$$\beta = -\frac{2A_0}{C} r_0, \quad \alpha = M\rho h A_0 r_0^2 - M\rho A_0 g_0 - \varkappa C r_0 \qquad (4.6)$$
$$k = (C^2 - A_0^2)/(A_0^2 C^2)$$
$$2m = -M^2\rho^2 hg - \frac{C^2}{A_0} M\rho h r_0^2 \left(1 + \frac{M\rho h}{A_0}\right)$$

we reduce (4.5) to the form

$$L = A_0^2 p^2 + 2\varkappa A_0 p\gamma + [-M\rho A_0 g - \varkappa C r_0]\gamma^2 + \qquad (4.7)$$
$$A_0^2 q^2 + 2\varkappa A_0 q\gamma + [-M\rho A_0 g - \varkappa C r_0]\gamma'^2 +$$
$$C\xi^2 + 2\varkappa' C\xi\delta + [-M\rho A_0 g - \varkappa' C r_0]\delta^2$$

$$\varkappa' = \varkappa + M\rho h r_0 C / A_0 \qquad (4.8)$$

Since according to (4.5) and because of $V_3 = 0$

$$\frac{dL}{dt} = mV_3\frac{d\delta}{dt} = 0 \qquad (4.9)$$

the sufficient condition of stability of unperturbed motion (4.1) is that the quadratic form (4.7) must be positive definite. This condition is ensured by the suitable selection of the, so far arbitrary, constant $\varkappa$. Actually each of the three quadratic forms which are additive in (4.7) are positive definite for any $\varkappa$ and $\varkappa'$ selected from the interval

$$\varkappa_1 \leqslant \varkappa \leqslant \varkappa_2, \quad \varkappa_1 \leqslant \varkappa' \leqslant \varkappa_2 \qquad (4.10)$$

$$\varkappa_{1,2} = -\frac{Cr_0}{2} \pm \left[\frac{C^2 r_0^2}{4} - M\rho A_0 g\right]^{1/2} \qquad (4.11)$$

The interval (4.10) actually exists when

$$C^2 r_0^2 > 4 M\rho A_0 g \qquad (4.12)$$

and inequality (4.12) by virtue of (4.8) together with the additional condition

$$M\rho h r_0 C / A_0 < \varkappa_2 - \varkappa_1 \qquad (4.13)$$

whose explicit form is

$$(1 - M^2\rho^2 h^2 / A_0^2) C^2 r_0^2 > 4M\rho A_0 g, \quad A_0 = M\rho h + A \qquad (4.14)$$

Obviously (4.12) holds when (4.14) is satisfied. Thus the single condition (4.14) is the sufficient condition of stability of the unperturbed motion (4.1). It is generalization of the sufficient condition of the Lagrange top stability, and reduces to it at the limit $h = 0$.

5. **The plane problem case.** The preceding analysis had disclosed a considerable variety of motions defined by Eq. (1.5), including the unlimited unwinding of the body and, also, bounded motions. Owing to the structure of Eq. (1.5) the coefficients at higher derivatives may pass through zero, and that generates shock phenomena, i.e. motions with unlimited angular velocity increase with limited angles. Singular motions, such as stationary and critical motions also exist.

All this appears in the plane problem considered below.

Setting

$$y - y_v = 0,\ z - z_v = h,\quad x - x_v = \sigma \tag{5.1}$$

we obtain from (1.5) the system of plane motion

$$\begin{gathered}[J + k_\rho h\cos\theta + k_\rho\,\sigma\sin\theta]\,\theta^{\cdot\cdot} + \\ k_\rho\,(\theta^{\cdot})^2\,[\sigma\cos\theta - h\sin\theta] - k_\rho g\sin\theta = Mg\sigma\end{gathered} \tag{5.2}$$

where $k_\rho = M\rho$ and J is the moment of inertia ($J = J_0 + M\rho^2$, where J_0 is the central moment of inertia) at the attachment point. Angle θ is measured from the positive direction of axis OZ relative to OX to the body axis of symmetry Oz' on which the body center of mass is assumed to lie.

Formula (5.2) shows, first of all, the existence of equilibrium positions defined by the equality

$$\sin\theta_0 = -\sigma/\rho \tag{5.3}$$

when condition

$$|\sigma/\rho| \leqslant 1 \tag{5.4}$$

is satisfied. When it is not satisfied there is no equilibrium.

Linearization of Eq. (5.2) around the equilibrium positions (5.3) enables us to ascertain that the "upper" equilibrium ($\cos\theta_0 > 0$) is unstable, while the "lower" ($\cos\theta_0 < 0$) is stable when condition

$$J_0 + M(\rho^2 - \sigma^2) > Mh\sqrt{\rho^2 - \sigma^2} \tag{5.5}$$

is satisfied and unstable in the opposite case.

Using the conventional substitution

$$\theta^{\cdot} = p,\quad \theta^{\cdot\cdot} = \frac{1}{2}\frac{d}{d\theta}p^2 \tag{5.6}$$

we reduce Eq. (5.2) to a linear inhomogeneous first order equation in p^2 with θ as the independent variable. That equation is integrable in quadratures. In this problem the quadratures are to be taken in explicit form. The result is of the form

$$\begin{aligned}&\theta' = [c - 2\cos\theta + \varkappa\theta + 2e\sigma\sin(\theta-\alpha)/\rho - e\cos(2\theta-\alpha)/2]^{1/2}\times \\ &\qquad [1 + e\cos(\theta-\alpha)]^{-1},\quad c = \text{const} \\ &\theta' = d\theta/d\tau,\quad \tau = \omega t,\quad \omega = \sqrt{M\rho g/J} \\ &\varkappa = \sigma(2/\rho + M\rho/J),\quad e = M\rho\sqrt{h^2+\sigma^2}/J \\ &\sin\alpha = \sigma/\sqrt{h^2+\sigma^2},\quad \cos\alpha = h/\sqrt{h^2+\sigma^2}\end{aligned} \tag{5.7}$$

This enables us to construct the pattern of motion in the plane θ, θ' in terms of four parameters: $\varkappa$, e, σ / ρ, and α.

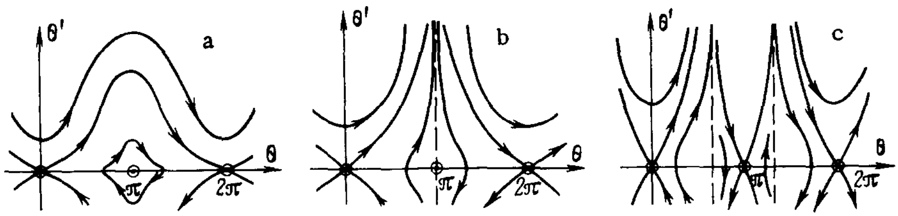

Fig. 3

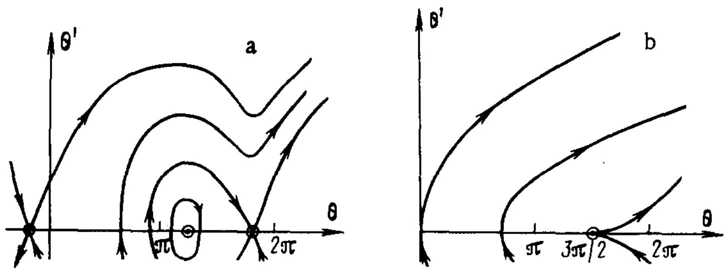

Fig. 4

The effect of parameter e is substantial. As implied by (5.7), the denominator in the right-hand side of (5.7) may pass through zero when $e \geqslant 1$, which implies an infinitely high angular velocity for finite values of the angle. We shall refer here to this phenomenon as "shock". The necessary and sufficient condition of absence of shock is $e < 1$.

The above is illustrated in Fig. 3, where the phase pattern relates to the case of $\sigma = 0$. When $e < 1$ the pattern is similar to that of the simple pendulum with the stable stationary point $\theta_0 = \pi$ and unstable $\theta = 0$ (Fig. 3, a).

As $e \to 1$, the oscillating motion region stretches more and more along the θ' -axis and, finally, for $e = 1$ becomes discontinuous; oscillating motions vanish and shock motions appears (Fig. 3, b), with the straight line $\theta_0 = \pi$ representing the discontinuity surface. Then, for $e > 1$ the discontinuity surface $\theta = \theta_*$ splits in two, each of which satisfies the condition $\cos \theta_* = -1 / e$, and point $\theta_0 = \pi$ becomes the second unstable point (Fig. 3c).

The described pattern corresponds to $\sigma = 0$, i.e. when the leg attachment point is exactly above the support point. When $\sigma \neq 0$ the pattern is complicated by the action of the constant moment and the presence of motion related to the infinite unwinding of the body. The phase patterns of that case appear in Fig. 4 with the condition of freedom from shock $e < 1$ satisfied. For definitenes we assume here that $\sigma > 0$. As shown above, when $\sigma < \rho$ there exists the stationary solution (5.3) and, also, a region of oscillatory motions in the neighborhood of the stable stationary motion regions. Trajectories of the body infinite unwinding also exist. As $\sigma \to \rho$, the region of oscillatory motions contracts, and finally vanishes when $\sigma = \rho$. Only

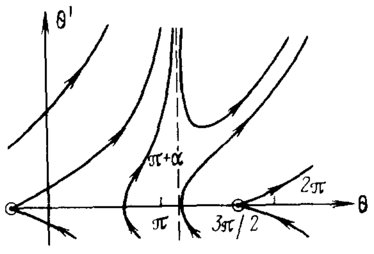

Fig. 5

the trajectories of infinite unwinding remain. They only become distorted when $\sigma > \rho$. Unlike in the case of $\sigma < \rho$, stationary solutions and critical motions are absent when $\sigma > \rho$.

It is not difficult to visualise phase trajectory characterized by, the absence of stationary points and the presence of shock phenomena, also when $\sigma \geqslant \rho$, $e \geqslant 1$. When σ and e are just slightly larger than ρ and unity, respectively, the phase trajectories shown in Fig. 4, b break at the discontinuity surfaces $\theta_* = \pi + \alpha$ (more exactly, when $\cos(\theta_* - \alpha) = -1 / e$), and approach infinity along the latter with respect to θ' (see Fig. 5, where $\sigma = \rho$, $e = 1$).

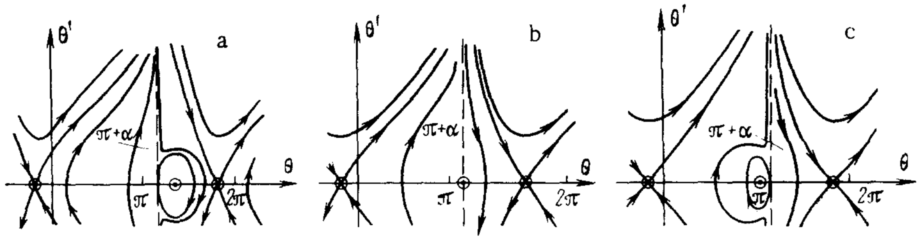

Fig. 6

The most complicated case occurs when conditions (5.4) of existence of stationary points and of shock $e \geqslant 1$ are simultaneously satisfied. The size of the region of oscillatory motions depends on how far one of the two stationary points $\theta_0 \in [\pi, 3\pi / 2]$ lies from the discontinuity line; when the discontinuity line passes through the stationary point θ_0 there are no oscillatory motions. Thus the necessary and sufficient conditions of existence of a region of limited motions are conditions (5.4) together with condition $\theta_0 \neq \theta^*$, where θ^* is determined by the equality $1 + e\cos(\theta^* - \alpha) = 0$ and condition $\theta^* \in [\pi, 3\pi / 2]$. Possible situations when $0 < \sigma < \rho$, and $e = 1$ are represented in Fig. 6, where [diagrams] a, b, and c, correspond, respectively, to the cases of $\theta_0 > \theta^*$, $\theta_0 = \theta^*$, and $\theta_0 < \theta^*$.

In the case of $\rho > \sigma \neq 0$, $e > 1$ the pattern is complicated by the presence of two discontinuity surfaces. The region of limited motions may lie to the left or right of these (the phase pattern can be obtained by studying Fig. 6) but not between them, since there the stability conditions (5.5) of stationary points are violated. In the latter case the phase pattern resembles that shown in Fig. 3, c.

The problem considered here corresponds to the limit case of the problem of pacing (with the step length and speed approaching zero). Hence the investigated motion can be considered as a generating one for pacing motions. They are also interesting on their own as an analysis of "conditions of standing".

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